2 Existence Theorems for Minimal Points

In this chapter we investigate a general optimization problem in a real normed space. For such a problem we present assumptions under which at least one minimal point exists. Moreover, we formulate simple statements on the set of minimal points. Finally the existence theorems obtained are applied to approximation and optimal control problems.

2.1 Problem Formulation

The standard assumption of this chapter reads as follows:

$$\left.\begin{array}{l}\text{Let } (X, \|\cdot\|) \text{ be a real normed space;}\\ \text{let } S \text{ be a nonempty subset of } X;\\ \text{and let } f : S \to \mathbb{R} \text{ be a given functional.}\end{array}\right\} \tag{2.1}$$

Under this assumption we investigate the optimization problem

$$\min_{x \in S} f(x), \tag{2.2}$$

i.e., we are looking for minimal points of f on S.

In general one does not know if the problem (2.2) makes sense because f does not need to have a minimal point on S. For instance, for $X = S = \mathbb{R}$ and $f(x) = e^x$ the optimization problem (2.2) is not solvable. In the next section we present conditions concerning f and S which ensure the solvability of the problem (2.2).

J. Jahn, *Introduction to the Theory of Nonlinear Optimization*,
https://doi.org/10.1007/978-3-030-42760-3_2

2.2 Existence Theorems

A known existence theorem is the Weierstraß theorem which says that every continuous function attains its minimum on a compact set. This statement is modified in such a way that useful existence theorems can be obtained for the general optimization problem (2.2).

Definition 2.1 (weakly lower semicontinuous functional).

Let the assumption (2.1) be satisfied. The functional f is called *weakly lower semicontinuous* if for every sequence $(x_n)_{n\in\mathbb{N}}$ in S converging weakly to some $\bar{x} \in S$ we have:

$$\liminf_{n\to\infty} f(x_n) \geq f(\bar{x})$$

(see Appendix A for the definition of the weak convergence).

Example 2.2 (weakly lower semicontinuous functional).

The functional $f : \mathbb{R} \to \mathbb{R}$ with

$$f(x) = \begin{Bmatrix} 0 \text{ if } x = 0 \\ 1 \text{ otherwise} \end{Bmatrix}$$

(see Fig. 2.1) is weakly lower semicontinuous (but not continuous at 0).

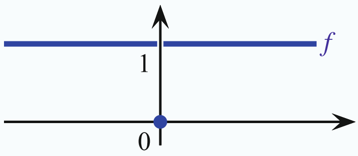

Fig. 2.1 Illustration of the functional f

Now we present the announced modification of the Weierstraß theorem.

Theorem 2.3 (solvability of problem (2.2)).

Let the assumption (2.1) be satisfied. If the set S is weakly sequentially compact and the functional f is weakly lower semicontinuous, then there is at least one $\bar{x} \in S$ with

$$f(\bar{x}) \leq f(x) \text{ for all } x \in S,$$

i.e., the optimization problem (2.2) has at least one solution.